

The asymptotics of the Touchard polynomials: a uniform approximation

R. B. PARIS

*Division of Computing and Mathematics,
University of Abertay Dundee, Dundee DD1 1HG, UK*

Abstract

The asymptotic expansion of the Touchard polynomials $T_n(z)$ (also known as the exponential polynomials) for large n and complex values of the variable z , where $|z|$ may be finite or allowed to be large like $O(n)$, has been recently considered in [4]. When $z = -x$ is negative, it is found that there is a coalescence of two contributory saddle points when $n/x = 1/e$. Here we determine the expansion when n and x satisfy this condition and also a uniform two-term approximation involving the Airy function in the neighbourhood of this value. Numerical results are given to illustrate the accuracy of the asymptotic approximations obtained.

Mathematics Subject Classification: 30E15, 33C45, 34E05, 41A30, 41A60

Keywords: Touchard polynomials, asymptotic expansion, method of steepest descents, uniform approximation

1. Introduction

The Touchard polynomials $T_n(z)$, also known as exponential polynomials, are defined by

$$T_n(z) = e^{-z} \sum_{k=0}^{\infty} \frac{k^n z^k}{k!} = e^{-z} \left(z \frac{d}{dz} \right)^n e^z \quad (1.1)$$

and were first introduced in a probabilistic context in 1939 by J. Touchard [5]. They have the generating function

$$\exp [z(e^t - 1)] = \sum_{n=0}^{\infty} T_n(z) \frac{t^n}{n!} \quad (1.2)$$

and possess the alternative representation given by

$$T_n(z) = \sum_{k=0}^n S(n, k) z^k, \quad (1.3)$$

where $S(n, k)$ is the Stirling number of the second kind [2, p. 624].

In [4] we considered the asymptotic expansion of $T_n(z)$ for large n and complex values of the variable z by an application of the method of steepest descents applied to a contour integral

representation. In this treatment $|z|$ was finite or allowed to be large like $O(n)$. It was found that there is an infinite number of saddle points of the integrand but that the precise number contributing to the expansion of $T_n(z)$ depended on the values of n and $|z|$. When $z = -x$ ($x > 0$), which is the central issue in this note, we have the expansions [4, Theorem 2]

$$T_{n-1}(-x) \sim \begin{cases} \Re \frac{\sqrt{2\Gamma(n)}e^{x+n/t_0}}{\sqrt{\pi(1+t_0)} t_0^{n-1}} \sum_{s=0}^{\infty} \frac{c_{2s}(t_0)\Gamma(s+\frac{1}{2})}{n^{s+\frac{1}{2}}\Gamma(\frac{1}{2})} & (\mu > 1/e) \\ \frac{\Gamma(n)e^{x+n/t_0}}{\sqrt{2\pi(1+t_0)} t_0^{n-1}} \sum_{s=0}^{\infty} \frac{c_{2s}(t_0)\Gamma(s+\frac{1}{2})}{n^{s+\frac{1}{2}}\Gamma(\frac{1}{2})} & (0 < \mu < 1/e) \end{cases} \quad (1.4)$$

as $n \rightarrow \infty$, where t_0 is one of the conjugate pair of roots of $te^t = -n/x$ with smallest modulus in the first expression and the smaller (negative) root in the second expression. Explicit expressions for the coefficients $c_{2s}(t_0)$ with $s \leq 2$ are given in [4]. In the case of the upper formula in (1.4), two conjugate saddles contribute to the expansion of $T_{n-1}(-x)$ when $1/e < \mu < \mu_1$, where $\mu_1 \doteq 3.11179$; when $\mu \geq \mu_1$, there are other conjugate pairs of contributory saddles but these are not included in the upper formula in (1.4) as they are subdominant as $n \rightarrow \infty$.

When $\mu := n/x = 1/e$, the two contributory saddle points coalesce to form a double saddle where the Poincaré-type expansions in (1.4) break down. In this note we obtain a uniform approximation for $T_{n-1}(-x)$ involving the Airy function together with an expansion valid when $\mu = 1/e$. Some numerical examples are given to illustrate the accuracy of the approximations obtained.

2. An integral representation

From (1.2) we obtain the integral representation

$$T_n(z) = \frac{n! e^{-z}}{2\pi i} \oint \frac{e^{ze^t}}{t^{n+1}} dt,$$

where the integration path is a closed circuit described in the positive sense surrounding the origin. We let $z = -x$, where the variable $x > 0$ is either finite or large like $O(n)$. Since $|\exp(-xe^t)| \rightarrow 0$ as $\Re(t) \rightarrow +\infty$ when $|\Im(t)| < \frac{1}{2}\pi$, it follows that the closed path above may be opened up into a loop¹, which commences at $+\infty$, encircles the origin and returns to $+\infty$. Then, introducing the scaled Touchard polynomial $\hat{T}_n(z)$ by

$$\hat{T}_n(z) \equiv \frac{1}{n!} T_n(z),$$

we have

$$\hat{T}_{n-1}(-x) = \frac{e^x}{2\pi i} \int_{\infty}^{(0+)} e^{n\psi(t)} dt, \quad (2.1)$$

where

$$\psi(t) \equiv \psi(t; \mu) := -\frac{e^t}{\mu} - \log t, \quad \mu := \frac{n}{x}. \quad (2.2)$$

Saddle points of the integrand occur when $\psi'(t) = 0$; that is when

$$te^t = -\mu,$$

¹In [4], where z is a complex variable and $n \geq 1$, the closed path around the origin was opened up into a loop which commences at $-\infty$, encircles the origin in the positive sense and returns to $-\infty$.

for which there is an infinite number of (complex) roots. For a full discussion of the distribution of the saddle points see [4]. When $0 < \mu < 1/e$, there are two saddles on the negative real axis given by the negative values of the Lambert- W function; see [2, p. 111]. When $\mu = 1/e$, these two saddles coalesce to form a double saddle point at $t = -1$ and when $\mu > 1/e$ the saddles move off the real axis to form a complex conjugate pair.

In Fig. 1 we show examples of the steepest paths through the contributory saddles when (i) $0 < \mu < 1/e$, (ii) $1/e < \mu < \mu_1$ and (iii) $\mu = 1/e$, where μ_1 is specified in Section 1. The t -plane has a branch cut along $[0, \infty)$. In case (i), the saddles t_0 and t_1 are situated on the negative real axis, with $t_0 \in (0, -1)$ and $t_1 \in (-1, -\infty)$ given by the negative branch of the Lambert- W function [2, p. 111]. The paths of steepest descent emanating from t_0 pass to $+\infty$ and the paths of steepest ascent from t_1 asymptotically approach the lines $\Im(t) = \pm\pi$ as $\Re(t) \rightarrow +\infty$. The integration path in (2.1) can then be deformed to pass over the steepest descent path emanating from t_0 . In case (ii), the saddles t_0 and t_1 form a conjugate pair and the integration path is the path labelled $ABCD$ in Fig. 1(b). When $\mu \geq \mu_1$, however, there are additional conjugate pairs of saddles (dependent on the value of μ) but these are subdominant as $n \rightarrow \infty$. In case (iii), the saddles t_0 and t_1 coalesce to form a double saddle at $t = -1$; the integration path then becomes the path CSB in Fig. 1(c).

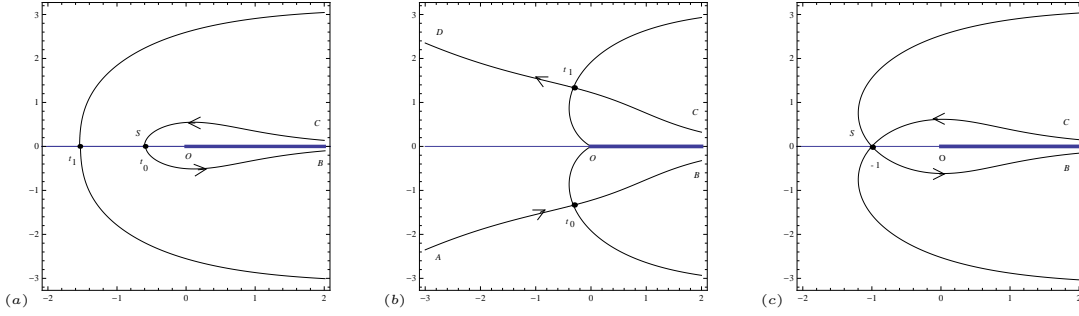


Figure 1: Paths of steepest descent and ascent through the saddles when (a) $0 < \mu < 1/e$, (b) $1/e < \mu < \mu_1$ and (c) $\mu = 1/e$. The saddles are denoted by heavy dots; the arrows indicate the direction of integration taken along steepest descent paths. There is a branch cut along $[0, \infty)$.

3. The asymptotics of $\hat{T}_{n-1}(-x)$ for $\mu \simeq 1/e$

Both the expansions in (1.4) cease to be valid in the neighbourhood of the double saddle at $t = -1$. We now determine an expansion valid at the coalescence point when $\mu = 1/e$ and a uniform two-term approximation when $\mu \simeq 1/e$.

3.1 The expansion of $\hat{T}_{n-1}(-x)$ when $\mu = 1/e$

When $\mu = 1/e$, the integration path can be deformed to coincide with the steepest descent path \mathcal{C} that enters $t = -1$ in the direction $\arg t = \pi/3$ and leaves to $t = -1$ in the direction $\arg t = -\pi/3$; see Fig. 1(c). If we put

$$-u = \psi(t) - \psi(-1) = \frac{\tau^3}{3!} + \frac{5\tau^4}{4!} + \frac{23\tau^5}{5!} + \frac{119\tau^6}{6!} + \frac{719\tau^7}{7!} + \dots, \quad \tau = t + 1$$

we find upon inversion using *Mathematica* that

$$\tau(w) = (6w)^{1/3} - \frac{5w^{2/3}}{2 \cdot 6^{1/3}} + \frac{33w}{40} - \frac{1463w^{4/3}}{720 \cdot 6^{2/3}} + \frac{126827w^{5/3}}{151200 \cdot 6^{1/3}} - \frac{15451w^2}{44800} + \dots,$$

where $w = e^{-\pi i}u$ on the path SB and $w = e^{\pi i}u$ on the path SC in Fig. 1(c). Then, upon differentiation of $\tau(w)$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-nu} \frac{d\tau}{du} du &= \frac{1}{2\pi i} \int_0^\infty e^{-nu} \left\{ \frac{d\tau(ue^{-\pi i})}{du} - \frac{d\tau(ue^{\pi i})}{du} \right\} du \\ &= \frac{1}{\pi} \int_0^\infty e^{-nu} \left\{ -\frac{6^{1/3} \sin \frac{1}{3}\pi}{3u^{2/3}} + \frac{5 \sin \frac{2}{3}\pi}{3 \cdot 6^{1/3} u^{1/3}} + \frac{1463 \sin \frac{4}{3}\pi}{540 \cdot 6^{2/3}} u^{1/3} + \dots \right\} du \\ &= -\frac{1}{3\pi} \left\{ \frac{\Gamma(\frac{1}{3}) \sin \frac{1}{3}\pi}{(\frac{1}{6}n)^{1/3}} - \frac{5\Gamma(\frac{2}{3}) \sin \frac{2}{3}\pi}{6(\frac{1}{6}n)^{2/3}} - \frac{1463\Gamma(\frac{4}{3}) \sin \frac{4}{3}\pi}{6480(\frac{1}{6}n)^{4/3}} - \dots \right\}. \end{aligned}$$

Hence we obtain the following result.

Theorem 1. *Let $\mu = n/x = 1/e$. Then as $n \rightarrow \infty$, we have the expansion for the scaled Touchard polynomial*

$$\hat{T}_{n-1}(-x) \sim (-)^{n-1} \frac{e^{x-n}}{3\pi} \sum_{m=0}^{\infty} \frac{(-)^m B_m \Gamma(\frac{1}{3}m + \frac{1}{3})}{(\frac{1}{6}n)^{(m+1)/3}} \sin \pi(\frac{1}{3}m + \frac{1}{3}), \quad (3.1)$$

where

$$B_0 = 1, \quad B_1 = \frac{5}{6}, \quad B_3 = \frac{1463}{6480}, \quad B_4 = \frac{126827}{1088640}, \quad B_6 = \frac{4732223}{167961600}, \dots$$

We note the omission of the coefficients with index $m = 2, 5, \dots$; these terms do not contribute to the expansion on account of the vanishing of the sine factor.

3.2 A uniform approximation for $\hat{T}_{n-1}(-x)$ when $\mu \simeq 1/e$

Let $\mu = 1/(e\xi)$ with $\xi > 0$; when $\xi \geq 1$ the saddles t_0 and t_1 are real, whereas when $\xi < 1$ the saddles form a conjugate pair. To obtain a uniform approximation valid for $\xi \simeq 1$ we apply the standard cubic transformation [1]

$$\psi(t) = \frac{1}{3}u^3 - \zeta u + \beta \quad (3.2)$$

to the integrand in (2.1). The quantities ζ and β depend on ξ and are determined by the requirement that the saddles t_0 and t_1 correspond to $u = \zeta^{1/2}$ and $u = -\zeta^{1/2}$, respectively; that is

$$\beta = \frac{1}{2}\{\psi(t_0) + \psi(t_1)\}, \quad \psi(t_j) = 1/t_j - \log t_j \quad (j = 0, 1) \quad (3.3)$$

and

$$\frac{2}{3}\zeta^{3/2} = \frac{1}{2}\{\psi(t_1) - \psi(t_0)\} \quad (\xi > 1), \quad \frac{2}{3}(-\zeta)^{3/2} = \frac{1}{2}i\{\psi(t_0) - \psi(t_1)\} \quad (\xi < 1). \quad (3.4)$$

For $x > 0$, the quantity $\zeta \geq 0$ when $\xi \geq 1$ and $\zeta < 0$ when $\xi < 1$.

The integral for $\hat{T}_{n-1}(-x)$ in (2.1) then becomes

$$\hat{T}_{n-1}(-x) = \frac{e^{x+n\beta}}{2\pi i} \int_{\mathcal{C}'} e^{n(\frac{1}{3}u^3 - \zeta u)} \frac{dt}{du} du,$$

where \mathcal{C}' is the image in the u -plane of the integration path. With the substitution

$$g(u) := \frac{dt}{du} = A_0 + B_0 + (u^2 - \zeta)G(u),$$

where

$$A_0 = \frac{1}{2}\{g(\zeta^{\frac{1}{2}}) + g(-\zeta^{\frac{1}{2}})\}, \quad B_0 = \frac{1}{2\zeta^{\frac{1}{2}}}\{g(\zeta^{\frac{1}{2}}) - g(-\zeta^{\frac{1}{2}})\}, \quad (3.5)$$

we then obtain [6, p. 369], [3, p. 67]

$$\hat{T}_{n-1}(-x) = e^{x+n\beta} \left\{ \frac{A_0}{n^{1/3}} U(n^{2/3}\zeta) - \frac{B_0}{n^{2/3}} U'(n^{2/3}\zeta) + \frac{n^{-1}}{2\pi i} \int_{\mathcal{C}} e^{n(\frac{1}{3}u^3 - \zeta u)} G'(u) du \right\}, \quad (3.6)$$

where the prime denotes differentiation with respect to the argument concerned and the function $U(z)$ is given by

$$U(z) = \frac{1}{2\pi i} \int_{\mathcal{C}'} e^{\frac{1}{3}\tau^3 - z\tau} d\tau.$$

The path \mathcal{C}' in the u -plane when $\xi > 1$ and $\xi < 1$ can be shown to be asymptotic to the rays $\arg u = \pm\pi/3$ traversed in the direction from the upper half-plane to the lower half-plane (we omit these details). From [2, Eq. (9.5.4)], the function $U(z)$ is therefore given by the Airy function $-\text{Ai}(z)$.

From [6, p. 367], we have

$$g(\zeta^{\frac{1}{2}}) = \left(\frac{2\zeta^{\frac{1}{2}}}{\psi''(t_0)} \right)^{1/2}, \quad g(-\zeta^{\frac{1}{2}}) = \left(\frac{-2\zeta^{\frac{1}{2}}}{\psi''(t_1)} \right)^{1/2} \quad (\zeta \neq 0),$$

where $\psi''(t_j) = (1 + t_j)/t_j^2$ ($j = 0, 1$). Then, from (3.5), we find

$$A_0 = \frac{\zeta^{\frac{1}{4}}}{\sqrt{2}} \left\{ \left(\frac{1}{\psi''(t_0)} \right)^{1/2} + \left(\frac{-1}{\psi''(t_1)} \right)^{1/2} \right\}, \quad B_0 = \frac{\zeta^{-\frac{1}{4}}}{\sqrt{2}} \left\{ \left(\frac{1}{\psi''(t_0)} \right)^{1/2} - \left(\frac{-1}{\psi''(t_1)} \right)^{1/2} \right\} \quad (3.7)$$

when $\xi > 1$, and

$$A_0 = \sqrt{2}|\zeta|^{\frac{1}{4}} \Re \left[\left(\frac{i}{\psi''(t_1)} \right)^{1/2} \right], \quad B_0 = \sqrt{2}|\zeta|^{-\frac{1}{4}} \Im \left[\left(\frac{i}{\psi''(t_1)} \right)^{1/2} \right] \quad (3.8)$$

when $\xi < 1$. Hence, upon neglecting the third term in braces in (3.6) (which is $o(n^{-1})$), we obtain the following result.

Theorem 2. *Let $\mu = n/x = 1/(e\xi)$, where $\xi > 0$. Then we have the uniform two-term approximation for the scaled Touchard polynomial*

$$\hat{T}_{n-1}(-x) \sim (-)^{n-1} e^{x+n\Re(\beta)} \left\{ \frac{A_0}{n^{1/3}} \text{Ai}(n^{2/3}\zeta) - \frac{B_0}{n^{2/3}} \text{Ai}'(n^{2/3}\zeta) \right\} \quad (3.9)$$

as $n \rightarrow \infty$. The quantities β and ζ are defined in (3.3) and (3.4), where $\zeta \geq 0$ for $\xi \geq 1$ and $\zeta < 0$ when $\xi < 1$. The coefficients A_0 and B_0 are given in (3.7) and (3.8).

At coalescence when $\xi = 1$ ($t = -1$, $u = 0$) we have $A_0 = g(0) = t'(0)$, $B_0 = g'(0) = (t'(0))^2 + t''(0)$, where $t(u) = dt/du$ and, by repeated differentiation of (3.2),

$$t'(0) = \left(\frac{2}{\psi'''(-1)} \right)^{1/3}, \quad t''(0) = -\frac{\psi^{iv}(-1)}{6\psi'''(-1)} \left(\frac{2}{\psi'''(-1)} \right)^{2/3}.$$

Since $\psi'''(-1) = 1$, $\psi^{iv}(-1) = 5$, we obtain $A_0 = 2^{1/3}$, $B_0 = -\frac{5}{6} \cdot 2^{2/3}$. Use of the standard values $\text{Ai}(0) = 3^{-2/3}/\Gamma(\frac{2}{3})$, $\text{Ai}'(0) = -3^{-1/3}/\Gamma(\frac{1}{3})$, together with $\psi(-1) = -1 - \pi i$ (so that $\Re(\beta) = -1$), then shows that the approximation (3.9) at coalescence becomes

$$\hat{T}_{n-1}(-x) \sim (-)^{n-1} \frac{e^{x-n}}{3\pi} \left\{ \frac{\Gamma(\frac{1}{3}) \sin \frac{1}{3}\pi}{(\frac{1}{6}n)^{1/3}} + \frac{5}{6} \frac{\Gamma(\frac{2}{3}) \sin \frac{2}{3}\pi}{(\frac{1}{6}n)^{2/3}} \right\} \quad (\mu = 1/e) \quad (3.10)$$

as $n \rightarrow \infty$. This is seen to agree with the first two terms of the expansion in (3.1).

3.3 Numerical examples

In Table 1 we illustrate the accuracy of the expansion (3.1) by presenting values of the absolute relative error in $\hat{T}_{n-1}(-x)$ for different values of n and truncation index m at the coalescence point $\mu = 1/e$. The value of $\hat{T}_{n-1}(-x)$ was computed from (1.3). Similarly, in Table 2, we show the absolute relative error in $\hat{T}_{n-1}(-x)$ for different values of the coalescence parameter ξ using the two-term approximation in (3.9) and, when $\xi = 1$, using (3.10).

Table 1: Values of absolute relative error in the computation of $\hat{T}_{n-1}(-x)$ using the asymptotic expansion (3.1) for different n and truncation index m when $\mu = 1/e$.

m	$n = 50$	$n = 80$	$n = 121$
0	2.514×10^{-1}	2.095×10^{-1}	1.788×10^{-1}
1	8.558×10^{-3}	5.390×10^{-3}	3.585×10^{-3}
3	2.744×10^{-3}	1.437×10^{-3}	8.144×10^{-4}
4	1.638×10^{-4}	6.490×10^{-5}	2.868×10^{-5}
6	6.184×10^{-5}	2.029×10^{-5}	7.616×10^{-6}

Table 2: Values of absolute relative error in the computation of $\hat{T}_{n-1}(-x)$ using the uniform approximation (3.9) and (3.10) for different values of the coalescence parameter ξ .

ξ	$n = 81$	$n = 100$	ξ	$n = 81$	$n = 100$
0.80	5.243×10^{-3}	8.179×10^{-3}	1.01	5.300×10^{-3}	4.301×10^{-3}
0.90	7.413×10^{-3}	3.322×10^{-3}	1.05	5.204×10^{-3}	4.222×10^{-3}
0.95	5.545×10^{-3}	4.540×10^{-3}	1.10	5.122×10^{-3}	4.153×10^{-3}
0.99	5.356×10^{-3}	4.355×10^{-3}	1.20	5.010×10^{-3}	4.060×10^{-3}
1.00	5.324×10^{-3}	4.326×10^{-3}	1.40	4.878×10^{-3}	3.951×10^{-3}

References

- [1] C. Chester, E. Friedman and F.J. Ursell, An extension of the method of steepest descents, *Proc. Camb. Phil. Soc.* **53** (1957) 599–611.
- [2] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [3] R.B. Paris, *Hadamard Expansions and Hyperasymptotic Evaluation*, Cambridge University Press, Cambridge, 2011.
- [4] R.B. Paris, The asymptotics of the Touchard polynomials, 2016. [arXiv:1606.07883].

- [5] J. Touchard, Sur les cycles des substitutions, Acta Math. **70**(1) (1939) 243–297.
- [6] R. Wong, *Asymptotic Expansion of Integrals*, Academic Press, London, 1989.